Conservation laws and the perturbed KdV equation

1990 J. Phys. A: Math. Gen. 23 4719

(http://iopscience.iop.org/0305-4470/23/20/033)

J. Phys. A: Math. Gen. 23 (1990) 4719-4724. Printed in the UK

ADDENDUM

Conservation laws and the perturbed KdV equation

Russell L Herman†

Department of Mathematics, St Lawrence University, Canton, NY 13617, USA

Received 15 May 1990

Abstract. In this addendum we examine the integral for the correction to the soliton position in the adiabatic approximation. Using the modified conservation laws for the mass and the energy, the general correction to the soliton position is obtained. It is easily seen that this correction is due to the formation of the shelf.

1. Introduction

The perturbation theory for the KdV equation is well known [3-6]. However, there have been several studies of the adiabatic effects of such perturbations on the soliton parameters, which arise from certain evolution equations for the associated discrete spectrum, using the modified conservation laws [4, 7]. In such studies the effects of the perturbation on the soliton amplitude are easily determined. However, the correction to the soliton velocity, or position, for a perturbed KdV soliton has been difficult to obtain in this way.

Using a full perturbation analysis, Karpman and Maslov have turned to an approximation to obtain the effects of the first-order correction [4]; while other studies, when corrected, have only resulted in the adiabatic result [2]. Namely, one begins with a perturbed KdV equation of the form

$$u_t + 6uu_x + u_{xxx} = \varepsilon P[u] \tag{1}$$

and assumes that the leading-order solitary wave is given by

$$u_0 = 2\eta^2 \operatorname{sech}^2 \eta(x - \bar{x}) \tag{2}$$

where the soliton parameters η and $\bar{x}$ depend on a slow time $\tau = \varepsilon t$. Then, the adiabatic velocity is found as [3, 4]

$$\bar{x}_\tau = 4\eta^2 + \frac{1}{4\eta^3}\int_{-\infty}^{\infty} P[u_0][\phi \operatorname{sech}^2 \phi + \tanh \phi]\, d\phi \tag{3}$$

while the non-adiabatic velocity is [3, 4]

$$\bar{x}_\tau = 4\eta^2 + \frac{1}{4\eta^3}\int_{-\infty}^{\infty} P[u_0][\phi \operatorname{sech}^2 \phi + \tanh \phi + \tanh^2 \phi]\, d\phi. \tag{4}$$

In this addendum we will show how the corrected integral in equation (4) can be obtained from that in equation (3). We will first use the conservation laws to obtain the leading-order expressions for η, the velocity, and the amplitude of the correction to the solitary wave. Using similar techniques, we then obtain the correction to the velocity in equation (4).

† Permanent address: Mathematical Sciences Department, University of North Carolina at Wilmington, Wilmington, NC 28403, USA.

0305-4470/90/204719+06$03.50

2. Conservation laws

It is well known that the KdV equation possess an infinite number of conserved quantities [1, 10]. We will be interested in modifying three conservation laws:

Conservation of mass/momentum

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} u\,\mathrm{d}x = 0 \tag{5}$$

Conservation of energy

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} u^2\,\mathrm{d}x = 0 \tag{6}$$

Conservation of the first moment

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} xu\,\mathrm{d}x = 3\int_{-\infty}^{\infty} u^2\,\mathrm{d}x = \text{constant} \tag{7}$$

for the perturbed KdV equation in (1), while paying close attention to the specific case of a damped KdV equation, which is given by

$$u_t + 6uu_x + u_{xxx} = -\Gamma u. \tag{8}$$

Integrating (1), we have the modified conservation of mass/momentum:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} u\,\mathrm{d}x = \varepsilon\int_{-\infty}^{\infty} P[u]\,\mathrm{d}x. \tag{9}$$

Multiplying equation (1) by u, and integrating:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} u^2\,\mathrm{d}x = 2\varepsilon\int_{-\infty}^{\infty} uP[u]\,\mathrm{d}x \tag{10}$$

which is the conservation of energy equation. Finally, we can multiply (1) by x and integrate to get the conservation equation for the first moment [8]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} xu\,\mathrm{d}x = 3\int_{-\infty}^{\infty} u^2\,\mathrm{d}x + \varepsilon\int_{-\infty}^{\infty} xP[u]\,\mathrm{d}x. \tag{11}$$

In many of the discussions on soliton perturbations researchers turn to these conservation laws to explain the effects of the disturbance on the initial soliton [4, 6, 7], or to check for the correctness of their results [5, 6]. Newell refers to this as 'the judicious use of the conservation laws' [11]. In fact, several researchers have relied on conservation laws to obtain the final results for the velocity correction [4, 9].

The standard argument for the damped KdV equation (8), first given by Kaup [5] and later rephrased [6, 11], goes as follows. If one assumes that u_0 is the dominant solution, then each side of the energy equation (10) yields to leading order

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} u^2\,\mathrm{d}x &\simeq -2\Gamma\int_{-\infty}^{\infty} u_0^2\,\mathrm{d}x = -\tfrac{32}{3}\eta^3\Gamma\\ \varepsilon\int_{-\infty}^{\infty} uP[u]\,\mathrm{d}x &\simeq -2\Gamma\int_{-\infty}^{\infty} u_0^2\,\mathrm{d}x = -\tfrac{32}{3}\eta^3\Gamma.\end{aligned} \tag{12}$$

Thus, to first order in Γ equation (10) holds. However, the mass equation (9) does not hold, since

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} u\,\mathrm{d}x &\simeq -\tfrac{2}{3}\Gamma\int_{-\infty}^{\infty} u_0\,\mathrm{d}x = -\tfrac{8}{3}\eta\Gamma\\ \varepsilon\int_{-\infty}^{\infty} P[u]\,\mathrm{d}x &\simeq -\Gamma\int_{-\infty}^{\infty} u_0\,\mathrm{d}x = -4\eta\Gamma.\end{aligned}\tag{13}$$

This mass discrepancy was first explained by Kaup [5]. Assuming that the solution to the perturbed equation (1) is given by the soliton (2) plus a correction,

$$u = u_0 + \varepsilon u_1$$

one finds that the correction behaves as [3]

$$\varepsilon u_1 \sim \begin{cases} -\Gamma/3\eta & 0<x<\bar{x}(t)\\ 0 & \text{otherwise.}\end{cases}\tag{14}$$

Even though the amplitude of this first-order solution is of order Γ, the area under this *shelf* is large enough to explain the mass difference. Computing the contribution on the left side of (9) due to the shelf, using u_0 in equation (2), u_1 from equation (14), and the first-order perturbation results [3],

$$\eta_t = -\tfrac{2}{3}\eta\Gamma \qquad \bar{x}_t = 4\eta^2 + O(\varepsilon),\tag{15}$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}\varepsilon u_1\,\mathrm{d}x \simeq \frac{\mathrm{d}}{\mathrm{d}t}\int_{0}^{\bar{x}(t)}\varepsilon u_1\,\mathrm{d}x = \varepsilon\bar{x}_t u_1(\bar{x}) \sim -\tfrac{4}{3}\eta\Gamma.\tag{16}$$

Thus, from equations (13)-(16), we find that the mass equation is balanced to first order:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}(u_0+\varepsilon u_1)\,\mathrm{d}x \simeq -\tfrac{8}{3}\eta\Gamma - \tfrac{4}{3}\eta\Gamma = -4\eta\Gamma \simeq \varepsilon\int_{-\infty}^{\infty} P[u]\,\mathrm{d}x.\tag{17}$$

Finally, using the same type of analysis one can show that (11) also holds to leading order. Again, in showing this one needs $\bar{x}_t = 4\eta^2$.

As noted by Kaup and Newell in other studies [6, 7, 11], the conservation laws can be used to derive the slow variations in the soliton parameters. Newell points out that the higher-order conservation laws do not give any new information [11]. In the next section we will use these conservation laws to obtain the results for the perturbed KdV equation (1). However, none of the conservation laws provides the first-order correction to the velocity in equation (4). In the last section we will use a similar method in order to obtain this correction.

3. General leading-order results

We begin with the conservation of mass equation (9). Letting $u = u_0 + \varepsilon u_1$, we have to leading order

$$\varepsilon\int_{-\infty}^{\infty} P[u_0]\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}(u_0+\varepsilon u_1)\,\mathrm{d}x + \mathrm{O}(\varepsilon^2).\tag{18}$$

We can bring the time derivative under the integral of the soliton part. The first-order solution consists, in general, of an oscillatory tail behind the soliton, a decaying part in front of the soliton and a possible shelf, such as we had seen in equation (14). The major contribution to the integral would come from the shelf, which we assume extends from $x=0$ to $x=\bar{x}(t)$. Assuming that the soliton parameters vary on a slow timescale $\tau=\varepsilon t$, the conservation equation can be approximated as

$$\varepsilon\int_{-\infty}^{\infty} P[u_0]\,\mathrm{d}x \simeq \varepsilon\int_{-\infty}^{\infty} u_{0_\tau}\,\mathrm{d}x+\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{\bar{x}} u_1\,\mathrm{d}x=\varepsilon\int_{-\infty}^{\infty} u_{0_\tau}\,\mathrm{d}x+\varepsilon\bar{x}_t u_1(\bar{x})+\mathrm{O}(\varepsilon^2) \qquad (19)$$

as in equation (16). Carrying out the integration over the soliton part gives

$$4\eta\eta_\tau=\int_{-\infty}^{\infty} P_1\,\mathrm{d}\phi-4\eta^3 u_1(\bar{x})+\mathrm{O}(\varepsilon) \qquad (20)$$

where we have used the leading-order velocity $\bar{x}_t=4\eta^2$. We note that this is just another form of the mass balance equation (16), which we had obtained for the damped case above.

A similar analysis can be carried out for the energy integral. In this case the first order correction, u_1, does not enter to leading order in ε. We have

$$\varepsilon\int_{-\infty}^{\infty} uP[u]\,\mathrm{d}x=\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} u^2\,\mathrm{d}x. \qquad (21)$$

Inserting the soliton part and integrating gives

$$4\eta\eta_\tau=\int_{-\infty}^{\infty} P[u_0]\,\mathrm{sech}^2\,\phi\,\mathrm{d}\phi+\mathrm{O}(\varepsilon). \qquad (22)$$

At this point we see that there are two expressions for $4\eta\eta_\tau$. The one given by (22) is the usual expression leading to the evolution of the soliton amplitude [3]. Subtracting this from (20) yields

$$4\eta^3 u_1(\bar{x})=\int_{-\infty}^{\infty} P[u_0]\tanh^2\phi\,\mathrm{d}\phi. \qquad (23)$$

Thus, we have obtained an expression for the amplitude of the shelf. For the damped KdV, we again find the shelf amplitude (see (14))

$$\varepsilon u_1(\bar{x})=\frac{1}{4\eta^3}\int_{-\infty}^{\infty}[-2\Gamma\eta^2\,\mathrm{sech}^2\,\phi]\tanh^2\phi\,\mathrm{d}\phi=-\frac{\Gamma}{3\eta}. \qquad (24)$$

We still do not have an expression for the time dependence of the phase. We can look at the third conservation law (11). Writing this out, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} xu\,\mathrm{d}x=\varepsilon\int_{-\infty}^{\infty} xP[u_0]\,\mathrm{d}x+3\int_{-\infty}^{\infty} u^2\,\mathrm{d}x=16\eta^3+\mathrm{O}(\varepsilon). \qquad (25)$$

Writing the phase as

$$\phi=\eta\left(x-\frac{1}{\varepsilon}x_0-x_1\right), \qquad (26)$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} xu\,\mathrm{d}x=\int_{-\infty}^{\infty}\frac{\mathrm{d}\phi}{\eta^2}\left[\phi+\eta\left(\frac{1}{\varepsilon}x_0+x_1\right)\right]u_{0_t}=4\eta x_{0_\tau}+\mathrm{O}(\varepsilon). \qquad (27)$$

The leading-order contribution gives

$$x_{0_\tau} = 4\eta^2. \tag{28}$$

(Note: this is consistent with the value used above for the leading-order velocity, $\bar{x}_t = 4\eta^2$.)

4. First-order correction to the velocity

We now turn to the first-order correction to the soliton velocity, which is given in equations (3) and (4). Both of these results contain the integral

$$\int_{-\infty}^{\infty} P[u][\phi \operatorname{sech}^2 \phi + \tanh \phi] \, \mathrm{d}\phi. \tag{29}$$

In parallel to the derivations of the above modified conservation laws, we multiply the perturbed KdV equation by $\phi \operatorname{sech}^2 \phi + \tanh \phi$ and integrate. Namely, we begin with

$$\int_{-\infty}^{\infty} P[u_0][\phi \operatorname{sech}^2 \phi + \tanh \phi] \, \mathrm{d}\phi$$

$$= \int_{-\infty}^{\infty} [u_t - 4\eta^3 u_\phi + 3\eta(u^2)_\phi + \eta^2 u_{\phi\phi\phi}][\tanh \phi + \phi \operatorname{sech}^2 \phi] \, \mathrm{d}\phi. \tag{30}$$

Integrating by parts and noting that u_1 is orthogonal to u_0 [3],

$$\int_{-\infty}^{\infty} u_1 \operatorname{sech}^2 \phi \, \mathrm{d}\phi = 0 \tag{31}$$

we have

$$4\eta^3 x_{1_\tau} \approx \int_{-\infty}^{\infty} \mathrm{d}\phi [\tanh \phi + \phi \operatorname{sech}^2 \phi] P_1 + 4\eta^3 u_1 \tanh \phi \,\Big|_{\bar{x}}^{\infty}. \tag{32}$$

Using the results (22), (23), and (28) from the above analyses, we obtain in the region for $\tanh^2 \phi \approx 1$,

$$4\eta^3 x_{1_\tau} \approx \int_{-\infty}^{\infty} \mathrm{d}\phi \, P[u_0][\phi \operatorname{sech}^2 \phi + \tanh \phi + \tanh^2 \phi]. \tag{33}$$

This result, combined with the leading-order velocity in equation (28), leads to the non-adiabatic value in equation (4).

5. Discussion

Summarizing these results, we have used the conservation laws plus an integration of the perturbed KdV equation, which was weighted by $\tanh \phi + \phi \operatorname{sech}^2 \phi$, to obtain the slow time variation of the soliton parameters and the shelf height. The correction to the adiabatic value in (3) is easily seen to be due to the presence of the shelf, through the use of equations (23) and (32). Furthermore, this is an asymptotic result, as we have approximated the $\tanh \phi$ in (32). We have shown in [2] that this result is supported by numerical experiment.

The use of the above weight is not as appealing as the use of conservation laws. At this point the integration in equation (30) is only suggested by the known results from some other direction; for example, there might be some other useful conservation law for the perturbed equation. To date the author has not found such a law. The fact that we must resort to such contortions to obtain useful information for an equation such as the perturbed KdV equation, leads us to be cautious when we make 'judicious use of conservation laws'.

References

[1] Gardner C S 1971 The Korteweg-de Vries equation and generalizations IV. The Korteweg-de Vries equation as a Hamiltonian system *J. Math. Phys.* **12** 1548-51

[2] Herman R L 1990 Resolution of the motion of a perturbed KdV soliton *Inverse Problems* **6** 43-54

[3] Herman R L 1990 A direct approach to studying soliton perturbations *J. Phys. A: Math. Gen.* **23** 2321-62

[4] Karpman V I and Maslov E M 1978 Structure of tails produced under the action of perturbations on solitons *Sov. Phys.-JETP* **48** 252-8

[5] Kaup D J 1977 Solitons as particles and the effects of perturbations *The Significance of Nonlinearity in the Natural Sciences* ed B Kursunoglu, A Perlmutter and L F Scott (New York: Plenum) pp 97-117

[6] Kaup D J and Newell A C 1978 Solitons as particles, oscillators, and in slowly changing media: a singular perturbation theory *Proc. R. Soc.* A **361** 413-6

[7] Kaup D J and Newell A C 1978 Theory of nonlinear oscillating dipolar excitations in one-dimensional condensates *Phys. Rev.* B **18** 5162-7

[8] Knickerbocker C J and Newell A C 1980 Shelves and the Korteweg-de Vries equation *J. Fluid Mech.* **98** 803-18

[9] Kodama Y and Ablowitz M J 1981 Perturbations of solitons and solitary waves *Stud. Appl. Math.* **64** 225-45

[10] Lax P 1968 Integrals of nonlinear evolution equations of evolution and solitary waves *Commun. Pure Appl. Math.* **21** 467-90

[11] Newell A C 1985 *Solitons in Mathematics and Physics* (Philadelphia, PA: SIAM)